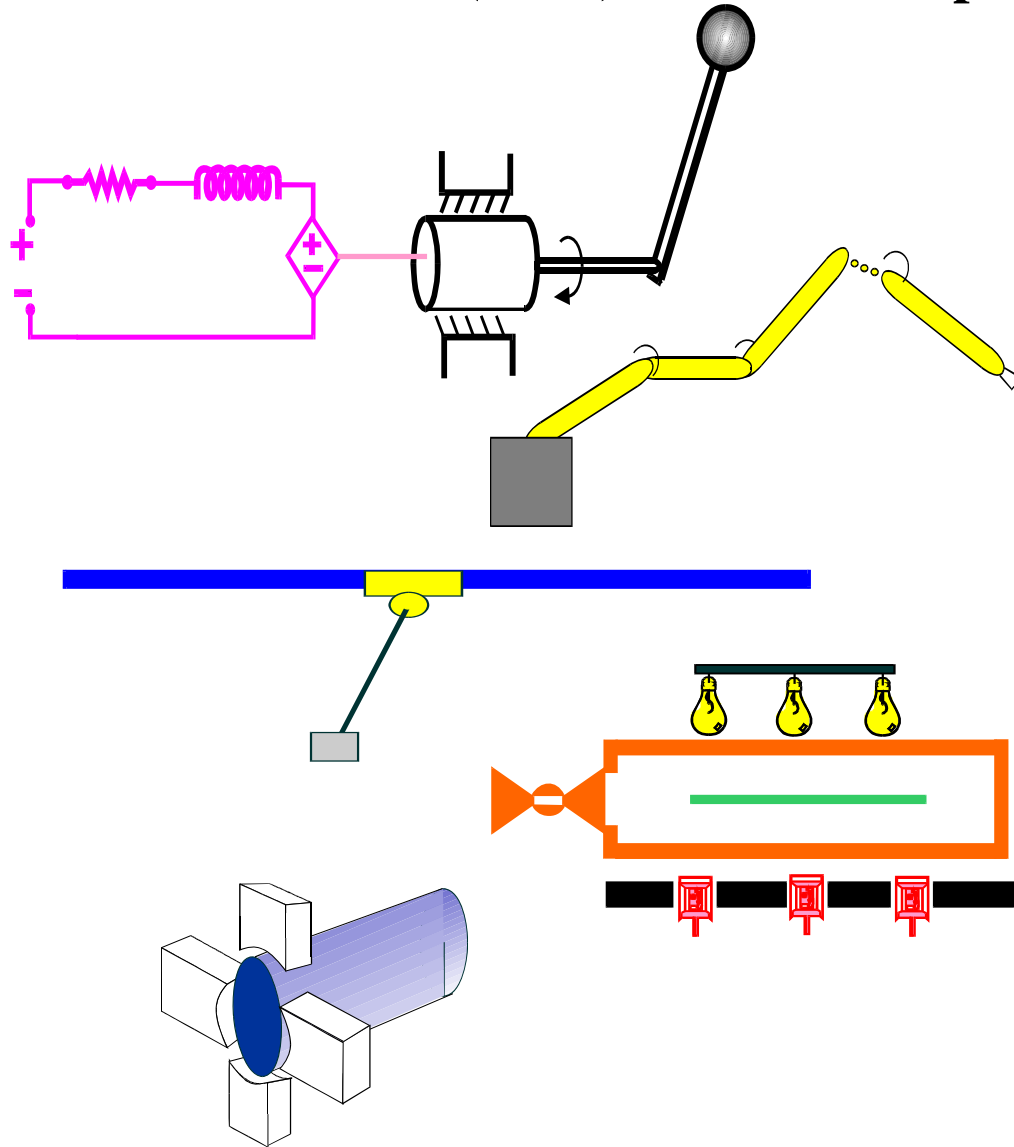


**Clemson University**  
**College of Engineering and Science**  
**Control and Robotics (CRB) Technical Report**



Number: CU/CRB/9/19/06/#2

Title: Adaptive Control of a Class of MIMO Nonlinear  
Systems in the Presence of Additive Input and Output  
Disturbances

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T. C. Burg

Report Documentation Page				Form Approved OMB No. 0704-0188	
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1. REPORT DATE <b>09 NOV 2009</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2009 to 00-00-2009</b>	
4. TITLE AND SUBTITLE <b>Adaptive Control of a Class of MIMO Nonlinear Systems in the Presence of Additive Input and Output Disturbances</b>				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Clemson University, Department of Electrical &amp; Computer Engineering, Clemson, SC, 29634-0915</b>				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT <b>see report</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>27</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

# Adaptive Control of a Class of MIMO Nonlinear Systems in the Presence of Additive Input and Output Disturbances<sup>★</sup>

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## Abstract

In this paper, two controllers are developed for a class of MIMO nonlinear systems. First, a robust adaptive controller is proposed and proven to yield semi-global asymptotic tracking in the presence of additive disturbances and parametric uncertainty. In addition to guaranteeing an asymptotic output tracking result, it is also proven that the parameter estimate vector is driven to a constant vector. In the second part of the paper, a learning controller is designed and proven to yield a semi-global asymptotic tracking result in the presence of additive disturbances when the desired trajectory is periodic. A continuous nonlinear integral feedback component is utilized in the design of both controllers and Lyapunov-based techniques are used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results are presented for both controllers.

*Key words:* Nonlinear systems; Adaptive control; Learning control; Disturbance rejection

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## 1 Introduction

It is the case of a class of MIMO nonlinear systems with parametric uncertainty and bounded disturbances that is considered here. Review of the basic control problem suggests and disqualifies certain solutions. It is probably wise at the outset to discard an exact model-based control approach for this problem given that any parameter estimation error and disturbances are not directly addressed, and hence, the system performance and stability cannot be predicted *a priori*. Given the parametric uncertainty in the proposed class of systems to be studied, an adaptive control solution may be warranted. However, an adaptive controller designed for a disturbance free system model may not compensate for the disturbances and may even go unstable under certain conditions. Enhancing the adaptive control approach with a robust component to form a robust adaptive controller can generally guarantee closed-loop signal boundedness in the presence of the additive disturbances. Unfortunately, while a robust adaptive controller can potentially guarantee the convergence of the tracking error to a bounded set (*i.e.*, the tracking error can't necessarily be driven to zero) the asymptotic tracking result (where the tracking error is driven to zero) that would be shown for an adaptive controller applied to the disturbance free model will be lost. These trade-offs in performance and robustness have framed the last ten years of research in robust adaptive control.

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<sup>★</sup> This paper was not presented at any IFAC meeting. A preliminary version of this paper appeared in [19]. Corresponding author E. Tatlicioglu. Tel. +90-232-7506728. Fax +90-232-7506599.

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Review of relevant work highlights some of the different tacks used to approach this problem. An adaptive backstepping controller was shown by Zhang and Ioannou in [23] for a class of single-input/single-output (SISO) linear systems with both input and output disturbances. The proposed controller demonstrates the use of a projection algorithm to bound the parameter estimates and guarantees an ultimately bounded tracking error. In an alternate approach, the work of Polycarpou and Ioannou [18] demonstrated a leakage-based adaptation law to compensate for parametric uncertainties. The proposed robust adaptive backstepping controller is applicable to a class of higher-order SISO systems with unknown nonlinearities. The suggested control law guarantees global uniform ultimate boundedness of the system state (with some restrictions on the bounding functions of the nonlinearities). Robust adaptive control laws were developed in [4], utilizing the modular design introduced in [11] and a tuning function design, for a class of systems similar to that studied in [18]. These authors show estimates on the effect of the bounded uncertainties and external disturbances on the tracking error. In [7], an adaptive backstepping controller for linear systems in the presence of output and multiplicative disturbances is designed. Ikhouane and Krstic, added a switching  $\sigma$ -modification to the tuning functions to obtain a tracking error proportional to the size of the perturbations. Marino and Tomei [15] proposed a robust adaptive tracking controller that achieves boundedness of all signals. The result is based on a class of SISO nonlinear systems that have additive disturbances but also unknown time-varying bounded parameters. It is significant that the result shows arbitrary disturbance attenuation. In [16], Pan and Basar proposed a robust adaptive controller for a similar class of systems in [15], where the tracking error is proven to be  $\mathcal{L}_2$ -bounded. In [5], Ge and Wang proposed a robust adaptive controller for SISO nonlinear systems with unknown parameters in the presence of disturbances, which ensure the global uniform boundedness of the tracking error.

Most of the research in adaptive control discussed above has focused on the convergence of the error signals and boundedness of the closed-loop system signals. As the sophistication in adaptive control techniques has evolved, additional questions about system performance have arisen. Notably, the final disposition of the parameters estimates in the closed-loop system has been examined. It is well established that without persistent excitation at the input, it is not typically possible to show the convergence of the parameter estimates to the corresponding system values (with an exception being a least-squares algorithm). In fact, for gradient and Lyapunov-type algorithms, convergence to a constant value, is typically not even guaranteed. Krstic summarized this question well in [10] and also provided some answers. In [10], it was shown that for the proposed adaptive controller; the parameter estimates will reach constant values after a sufficient amount of time.

A recent paper by Cai *et al.* [2]<sup>1</sup> presented a robust adaptive controller for MIMO nonlinear systems with parametric uncertainty and additive disturbances. It was assumed that the disturbance is twice continuously differentiable and has bounded time derivatives up to second order, the proposed controller was proven to yield an asymptotic output tracking result. However, no mention of the convergence of the parameter estimates was made. Thinking out loud for a moment, it *might* stand to reason that if the robust part of the controller is compensating for the disturbances and an asymptotic tracking result is obtained then perhaps something special is happening to the parameter estimates. Exploring this vague notion with mathematical rigor, we will show that with a minor modification to the control in [2] and with some additional analysis of the stability result, we are able to formulate a new conclusion about the parameter estimates. What is shown is that this robust adaptive controller will yield constant parameter estimates even in the presence of the disturbance. The stability analysis parallels that presented in [2] but with the extended analysis the convergence of the parameter estimates is demonstrated. In the design of the adaptive controller, the robust control component in [20] was combined with an adaptive control design to achieve semi-global asymptotic tracking. One contribution of this paper is to add to the small number of results where parameter convergence has been shown. In the second part of the paper, a learning controller for the same class of MIMO nonlinear systems is designed under the assumption that the reference trajectory is periodic (for past research related to the design of learning controllers, reader is referred to [1], [6], [22] and the references therein). In the design of the learning controller, the robust control component in [20] was combined with a nonlinear learning control design to compensate for the unknown system dynamics and a semi-global asymptotic tracking result is obtained in the presence of bounded additive input and output disturbances. When compared to [20], the two control methods developed in this paper require less control energy. The adaptive controller includes estimates for the unknown system parameters, and the learning control design embeds a learning component to compensate for the uncertain system dynamics when the desired trajectory is periodic. In both control designs, Lyapunov-based techniques are used to guarantee that the tracking error is asymptotically driven to zero. Numerical simulation results are presented for both controllers to demonstrate their viability.

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<sup>1</sup> [2] is the technical report version of [3].

## 2 Adaptive Control

### 2.1 Problem Statement

Following class of MIMO nonlinear systems is considered

$$\dot{x}^{(n)} = f + G(u + d_1) + d_2 \quad (1)$$

where  $x^{(i)}(t) \in \mathbb{R}^m$ ,  $i = 0, \dots, (n-1)$ , are the system states,  $f(x, \dot{x}, \dots, x^{(n-1)}, \theta) \in \mathbb{R}^m$ ,  $G(x, \dot{x}, \dots, x^{(n-1)}, \theta) \in \mathbb{R}^{m \times m}$  are nonlinear functions,  $\theta \in \mathbb{R}^p$  is an unknown constant parameter vector,  $d_1(t), d_2(t) \in \mathbb{R}^m$  are unknown additive nonlinear disturbances, and  $u(t) \in \mathbb{R}^m$  is the control input. The system model can be rewritten

$$Mx^{(n)} = h + u + d_1 + Md_2 \quad (2)$$

where  $M(x, \dot{x}, \dots, x^{(n-1)}, \theta) \in \mathbb{R}^{m \times m}$  and  $h(t) \in \mathbb{R}^m$  are defined as

$$M \triangleq G^{-1}, \quad (3)$$

$$h \triangleq Mf. \quad (4)$$

The system model is assumed to satisfy the following assumptions.

**Assumption 1** The nonlinear function  $G(\cdot)$  is symmetric, positive definite, and its inverse  $M(\cdot)$  satisfies the inequalities[12]

$$\underline{m} \|\xi\|^2 \leq \xi^T M(\cdot) \xi \leq \bar{m}(\cdot) \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^m \quad (5)$$

$\bar{m}(x, \dot{x}, \dots, x^{(n-1)}) \in \mathbb{R}$  is a positive, globally invertible, nondecreasing function of each variable,  $\underline{m} \in \mathbb{R}$  is a positive bounding constant, and  $\|\cdot\|$  denotes the Euclidean norm. This assumption holds for most electromechanical systems.

**Assumption 2** The nonlinear functions,  $f(\cdot)$  and  $G(\cdot)$ , are continuously differentiable up to their second derivatives (i.e.,  $f(\cdot), G(\cdot) \in \mathcal{C}^2$ ).

**Assumption 3** The nonlinear functions<sup>2</sup>,  $f(\cdot)$  and  $M(\cdot)$ , are affine in  $\theta$ .

**Assumption 4** The additive disturbances,  $d_1(t)$  and  $d_2(t)$ , are assumed to be continuously differentiable and bounded up to their second derivatives (i.e.,  $d_i(t) \in \mathcal{C}^2$  and  $d_i(t), \dot{d}_i(t), \ddot{d}_i(t) \in \mathcal{L}_\infty$ ,  $i = 1, 2$ ).

The output tracking error  $e_1(t) \in \mathbb{R}^m$  is defined as

$$e_1 \triangleq x_r - x \quad (6)$$

where  $x_r(t) \in \mathbb{R}^m$  is the reference trajectory satisfying the following property

$$x_r(t) \in \mathcal{C}^n, \quad x_r^{(i)}(t) \in \mathcal{L}_\infty, \quad i = 0, 1, \dots, (n+2). \quad (7)$$

The control design objective is to develop an adaptive control law that ensures  $\|e_1^{(i)}(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ ,  $i = 0, \dots, (n-1)$ , and that all signals remain bounded within the closed-loop system. To achieve the control objectives, the subsequent development is derived based on the assumption that the system states  $x^{(i)}(t)$ ,  $i = 0, \dots, (n-1)$  are measurable.

<sup>2</sup> When considering mechatronic systems one main concern in regards to this assumption is the existence of friction. The reader is referred to [14] and [17] for adaptive controllers that considered a nonlinear parameterizable friction model which was introduced in [13]. The control design developed in this paper can also compensate for this type of nonlinear friction by following the similar steps in [17].

## 2.2 Development of Robust Adaptive Control Law

The filtered tracking error signals,  $e_i(t) \in \mathbb{R}^m$ ,  $i = 2, 3, \dots, n$  are defined as follows

$$e_2 \triangleq \dot{e}_1 + e_1 \quad (8a)$$

$$e_3 \triangleq \dot{e}_2 + e_2 + e_1 \quad (8b)$$

$$\vdots$$

$$e_n \triangleq \dot{e}_{n-1} + e_{n-1} + e_{n-2}. \quad (8c)$$

A general expression for  $e_i$ ,  $i = 2, 3, \dots, n$  in terms of  $e_1$  and its time derivatives is given as

$$e_i = \sum_{j=0}^{i-1} a_{i,j} \dot{e}_1^{(j)} \quad (9)$$

where the known constant coefficients  $a_{i,j}$  are generated via a Fibonacci number series<sup>3</sup> [20], [21]. To facilitate the control development, the filtered tracking error signal, denoted by  $r(t) \in \mathbb{R}^m$ , is defined by

$$r \triangleq \dot{e}_n + \Lambda e_n \quad (10)$$

in which  $\Lambda \in \mathbb{R}^{m \times m}$  is a constant, diagonal, positive definite, gain matrix. By differentiating (10) and premultiplying by  $M(\cdot)$ , the following expression can be derived<sup>4</sup>

$$\begin{aligned} M\dot{r} = & M \left( \dot{x}_r^{(n+1)} + \sum_{j=0}^{n-2} a_{nj} \dot{e}_1^{(j+2)} + \Lambda \dot{e}_n \right) \\ & + \dot{M}x^{(n)} - \dot{h} - \dot{u} - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 \end{aligned} \quad (11)$$

note that (6), (9), the first time derivative of (2), and the fact that  $a_{n,(n-1)} = 1$  were utilized. The expression in (11) can be arranged as follows

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_n - \dot{u} + N - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 \quad (12)$$

where the auxiliary function  $N(x, \dot{x}, \dots, x^{(n)}, t) \in \mathbb{R}^m$  was introduced with the definition

$$\begin{aligned} N \triangleq & M \left( \dot{x}_r^{(n+1)} + \sum_{j=0}^{n-2} a_{nj} \dot{e}_1^{(j+2)} + \Lambda \dot{e}_n \right) \\ & + \dot{M} \left( x^{(n)} + \frac{1}{2}r \right) + e_n - \dot{h}. \end{aligned} \quad (13)$$

To facilitate the subsequent analysis, (12) can be rearranged as

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_n - \dot{u} + \tilde{N} + N_r + \psi \quad (14)$$

<sup>3</sup> By definition, the first two Fibonacci numbers are 0 and 1, and each remaining number is the sum of the previous two [9].

<sup>4</sup> The open-loop error system in (11) is developed in detail for a second order system in Appendix D.

where  $\tilde{N}(x, \dot{x}, \dots, x^{(n)}, t), N_r(t), \psi(t) \in \mathbb{R}^m$  are defined

$$\tilde{N} \triangleq (N - M\dot{d}_2 - \dot{M}d_2) - (N_r - M_r\dot{d}_2 - \dot{M}_rd_2) \quad (15)$$

$$N_r \triangleq N|_{x=x_r, \dot{x}=\dot{x}_r, \dots, x^{(n)}=x_r^{(n)}} \quad (16)$$

$$\psi \triangleq -\dot{d}_1 - M_r\dot{d}_2 - \dot{M}_rd_2 \quad (17)$$

in which  $M_r(t) \in \mathbb{R}^{m \times m}$  represents

$$M_r \triangleq M|_{x=x_r, \dot{x}=\dot{x}_r, \dots, x^{(n-1)}=x_r^{(n-1)}}. \quad (18)$$

**Remark 1** By utilizing the Mean Value Theorem along with Assumptions 2 and 4, the following upper bound can be developed<sup>5</sup>

$$\|\tilde{N}(\cdot)\| \leq \rho(\|z\|)\|z\| \quad (19)$$

where  $z(t) \in \mathbb{R}^{(n+1)m \times 1}$  is defined by

$$z \triangleq [e_1^T \ e_2^T \ \dots \ e_n^T \ r^T]^T \quad (20)$$

and  $\rho(\cdot) \in \mathbb{R}_{\geq 0}$  is some globally invertible, nondecreasing function.

**Remark 2** It is clear from (7), Assumption 4, (17), and the time derivative of (17) that  $\psi(t), \dot{\psi}(t) \in \mathcal{L}_\infty$ .

**Remark 3** It can be seen from (7), (13), (16), and the time derivative of (16) that  $N_r(t), \dot{N}_r(t) \in \mathcal{L}_\infty$ .

**Remark 4** In view of Assumption 3,  $N_r(\cdot)$  defined in (16) can be linearly parameterized in the sense that

$$N_r = W_r \theta \quad (21)$$

where  $W_r(t) \in \mathbb{R}^{m \times p}$  is the known regressor matrix and is a function of only  $x_r(t)$  and its time derivatives.

Based on (14) and (21), the control input is designed as

$$\begin{aligned} u = & (K + I_m) \left[ e_n(t) - e_n(t_0) + \Lambda \int_{t_0}^t e_n(\tau) d\tau \right] \\ & + \int_{t_0}^t W_r(\tau) \hat{\theta}(\tau) d\tau + \Pi \end{aligned} \quad (22)$$

where the auxiliary signal  $\Pi(t) \in \mathbb{R}^m$  is generated according to the following update law

$$\dot{\Pi} = (C_1 + C_2) \text{Sgn}(e_n), \Pi(t_0) = 0_{m \times 1} \quad (23)$$

where  $\hat{\theta}(t) \in \mathbb{R}^p$  denotes the parameter estimate vector and is generated via

$$\begin{aligned} \hat{\theta} = & \Gamma \int_{t_0}^t W_r^T(\tau) \Lambda e_n(\tau) d\tau - \Gamma \int_{t_0}^t \dot{W}_r^T(\tau) e_n(\tau) d\tau \\ & + \Gamma W_r^T(t) e_n(t) - \Gamma W_r^T(t_0) e_n(t_0). \end{aligned} \quad (24)$$

In (22)-(24),  $K, C_1, C_2 \in \mathbb{R}^{m \times m}$  and  $\Gamma \in \mathbb{R}^{p \times p}$  are constant, diagonal, positive definite, gain matrices,  $I_m \in \mathbb{R}^{m \times m}$  is the standard identity matrix, and  $\text{Sgn}(\cdot) \in \mathbb{R}^m$  being the vector signum function. It should be noted that  $\hat{\theta}(t_0) = 0_{p \times 1}$

<sup>5</sup> The reader is referred to Appendix F for the derivation of the upper bound in (19).

and  $u(t_0) = 0_{m \times 1}$  where  $0_{p \times 1} \in \mathbb{R}^p$  and  $0_{m \times 1} \in \mathbb{R}^m$  are vectors of zeros. Based on the structure of (22)-(23), the following are obtained<sup>6</sup>

$$\dot{u} = (K + I_m) r + (C_1 + C_2) \text{Sgn}(e_n) + W_r \hat{\theta} \quad (25)$$

$$\dot{\hat{\theta}} = \Gamma W_r^T r. \quad (26)$$

Finally, after substituting (25) into (14), the following closed-loop error system for  $r(t)$  is obtained

$$\begin{aligned} M\dot{r} = & -\frac{1}{2}\dot{M}r - e_n - (K + I_m)r + W_r\tilde{\theta} \\ & - (C_1 + C_2) \text{Sgn}(e_n) + \tilde{N} + \psi \end{aligned} \quad (27)$$

where the parameter estimation error signal  $\tilde{\theta}(t) \in \mathbb{R}^p$  is defined as follows

$$\tilde{\theta} \triangleq \theta - \hat{\theta}. \quad (28)$$

### 2.3 Stability Analysis

**Theorem 1** *The control law (22), (23) and the update law (24) ensure the boundedness of all closed-loop system signals and  $\|e_1^{(i)}(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ ,  $i = 0, \dots, n$ , provided*

$$\lambda_{\min}(\Lambda) > \frac{1}{2}, \quad (29)$$

$$C_{1i} > \|\psi_i(t)\|_{\mathcal{L}_\infty} + \frac{1}{\Lambda_i} \|\dot{\psi}_i(t)\|_{\mathcal{L}_\infty} \quad (30)$$

where the subscript  $i = 1, \dots, m$  denotes the  $i$ th element of the vector or diagonal matrix and the elements of  $K$  are selected sufficiently large relative to the system initial conditions (see Appendix A for proof).

**Theorem 2** *There exists a constant vector  $\hat{\theta}_\infty \in \mathbb{R}^p$  such that*

$$\hat{\theta}(t) \rightarrow \hat{\theta}_\infty \text{ as } t \rightarrow \infty \quad (31)$$

(see Appendix B for proof).

### 2.4 Numerical Simulation Results

A numerical simulation was performed to demonstrate the performance of the adaptive controller given in (22)-(24). A first order form of the class of systems considered in this paper<sup>7</sup> with the following modelling functions is utilized [2]

$$\begin{aligned} f &= \begin{bmatrix} x_1 x_2 \\ x_2^2 \end{bmatrix}, \quad G = \begin{bmatrix} \frac{2 + \cos x_1}{\theta_1} & 0 \\ 0 & \frac{3 + \sin x_2}{\theta_2} \end{bmatrix}, \\ \theta &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} \cos(2t) + \exp(-0.5t) \\ \sin(3t) + \exp(-0.5t) \end{bmatrix} \\ & \quad d_2 = \begin{bmatrix} \sin(2t) + \exp(-0.5t) \\ \cos(3t) + \exp(-0.5t) \end{bmatrix} \end{aligned}$$

<sup>6</sup> The expressions in (22), (23) and (25) are based on [20] and [21].

<sup>7</sup> The system model utilized in this simulation study is presented in detail in Appendix E.



where  $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ . The reference trajectory was selected as

$$x_r = \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} = \begin{bmatrix} \sin t \left( 1 - \exp \left( -\frac{t^3}{5} \right) \right) \\ \cos t \left( 1 - \exp \left( -\frac{t^3}{2} \right) \right) \end{bmatrix}.$$

The initial conditions of the system were set to  $x(t_0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$  and  $\hat{\theta}(t_0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ , while the controller parameters were chosen as  $\Lambda = 4I_2$ ,  $K = 2I_2$ ,  $C_1 = 2I_2$ ,  $C_2 = 2I_2$ , and  $\Gamma = 200I_2$ . The controller parameters above were selected via a trial-error method until a good tracking performance was obtained and then the lower control gain value was preferred. The tracking error  $e_1(t)$  is presented in Figure 1 where it is clear that the tracking objective is satisfied. In Figures 2 and 3, the parameter estimate  $\hat{\theta}(t)$  and the control input  $u(t)$  are presented, respectively. From Figure 2, it is clear that the parameter estimate vector is driven to a constant vector.

To demonstrate the effect of the adaptive term  $\int_{t_0}^t W_r(\tau) \hat{\theta}(\tau) d\tau$  in the adaptive controller in (22), during the simulation run following performance measures were computed

$$M_e(t) \triangleq \int_{t_0}^t \|e_1(\tau)\|^2 d\tau \quad (32)$$

$$M_u(t) \triangleq \int_{t_0}^t \|u(\tau)\|^2 d\tau \quad (33)$$

where  $M_e(t)$  is a measure of the magnitude of the tracking error, and  $M_u(t)$  is a measure of the energy expended by the controller over a period of the operation of the system. For both runs, it was observed that the tracking error converged to zero within 4 seconds. From Table 1, it is clear that after adding the adaptive term, the controller required less energy while achieving improved tracking performance.

Table 1  
Comparison of Energy Measures for Adaptive Control

	$u(t)$ without $\int_{t_0}^t W_r(\tau) \hat{\theta}(\tau) d\tau$	$u(t)$ with $\int_{t_0}^t W_r(\tau) \hat{\theta}(\tau) d\tau$
$M_e$	1.5559	1.4611
$M_u$	49.1676	47.8269

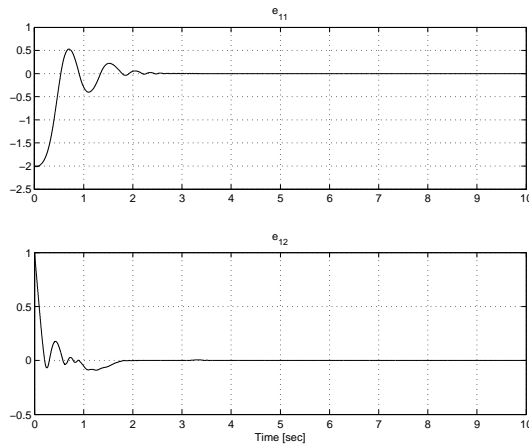


Fig. 1. (Adaptive Controller) Tracking Error  $e_1(t)$

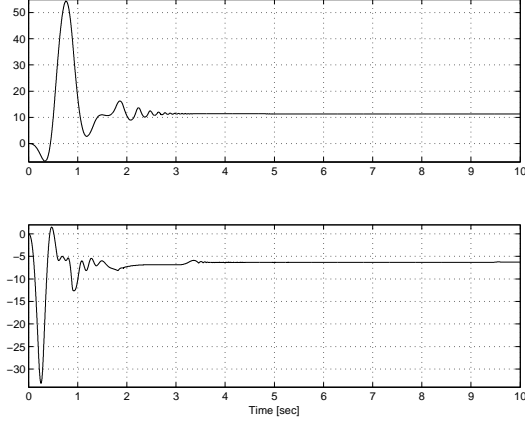


Fig. 2. (Adaptive Controller) Parameter Estimate  $\hat{\theta}(t)$  (top plot is  $\hat{\theta}_1(t)$  and the bottom one is  $\hat{\theta}_2(t)$ )

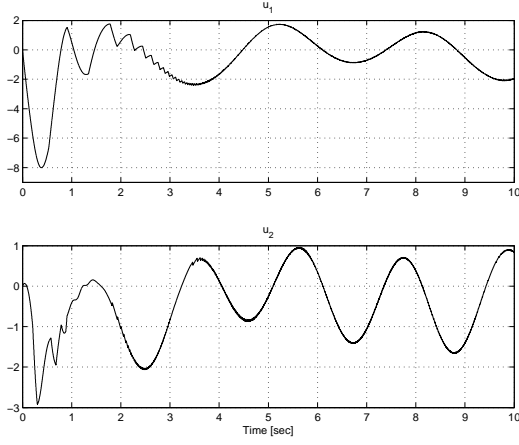


Fig. 3. (Adaptive Controller) Control Input  $u(t)$

### 3 Learning Control

#### 3.1 Problem Statement

The system model in (1) is considered. The nonlinear functions  $f(x, \dot{x}, \dots, x^{(n-1)}) \in \mathbb{R}^m$  and  $G(x, \dot{x}, \dots, x^{(n-1)}) \in \mathbb{R}^{m \times m}$  are uncertain where this dynamic uncertainty is assumed to be non-parameterizable. The system model is assumed to satisfy Assumptions 1, 2, and 4.

The output tracking error  $e_1(t)$  is defined in (6) and in this case the reference trajectory is periodic in the sense that

$$x_r^{(i)}(t+T) = x_r^{(i)}(t), \quad x_r^{(i)}(t) \in \mathcal{L}_\infty, i = 0, \dots, (n+2) \quad (34)$$

where  $T \in \mathbb{R}^+$  is the period of the reference trajectory.

The control design objective is to develop a nonlinear control law that ensures  $\|e_1(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ . To achieve the control objective, the subsequent development is derived based on the assumption that the system states  $x^{(i)}(t)$ ,  $i = 0, \dots, (n-1)$  are measurable.

### 3.2 Development of Learning Control Law

The open-loop error system development for the learning control law is exactly the same as the open-loop error system development for the adaptive control law. The control design is assumed to continue after Remark 1.

**Remark 5** It can be deduced from (34), Assumption 4, (17), and the time derivative of (17) that  $\psi(t), \dot{\psi}(t) \in \mathcal{L}_\infty$ .

**Remark 6** It can be seen from (13), (16), (34), and the time derivative of (16) that  $N_r(t), \dot{N}_r(t) \in \mathcal{L}_\infty$ .

**Remark 7** After utilizing (34), it is clear that  $N_r(t)$  satisfies the following equation

$$N_r(t+T) = N_r(t). \quad (35)$$

Based on (14), the control input is designed as

$$\begin{aligned} u = & (K + I_m) e_n(t) - (K + I_m) e_n(t_0) + \hat{W}_r(t) \\ & + \int_{t_0}^t (K + I_m) \Lambda e_n(\tau) d\tau + \Pi \end{aligned} \quad (36)$$

where the auxiliary signal  $\Pi(t) \in \mathbb{R}^m$  is generated according to the following update law

$$\dot{\Pi} = C_1 \text{Sgn}(e_n), \Pi(t_0) = 0_{m \times 1} \quad (37)$$

and  $\hat{W}_r(t) \in \mathbb{R}^m$  is defined as follows

$$\begin{aligned} \hat{W}_r(t) = & \hat{W}_r(t-T) + k_L \Lambda \int_{t_0}^t e_n(\tau) d\tau \\ & + k_L e_n(t) - k_L e_n(t_0). \end{aligned} \quad (38)$$

In (36)-(38),  $K, C_1, \Lambda \in \mathbb{R}^{m \times m}$  are constant, diagonal, positive definite, gain matrices and  $k_L \in \mathbb{R}$  is a positive gain. It should be noted that since  $\hat{W}_r(t_0) = 0_{m \times 1}$  it follows that  $u(t_0) = 0_{m \times 1}$ . The auxiliary function  $\hat{N}_r(t) \in \mathbb{R}^m$  is defined as

$$\hat{N}_r \triangleq \hat{W}_r. \quad (39)$$

By utilizing (39) along with (38), the following can be obtained

$$\hat{N}_r(t) = \hat{N}_r(t-T) + k_L r(t). \quad (40)$$

Taking the time derivative of (36) and substituting from (37) and (39) generates<sup>8</sup>

$$\dot{u} = (K + I_m) r + C_1 \text{Sgn}(e_n) + \hat{N}_r(t). \quad (41)$$

Finally, after substituting (41) into (14), the closed-loop error system for  $r(t)$  is obtained as follows

$$\begin{aligned} M\dot{r} = & -\frac{1}{2}\dot{M}r - e_n - (K + I_m)r \\ & - C_1 \text{Sgn}(e_n) + \tilde{N} + \tilde{N}_r + \psi \end{aligned} \quad (42)$$

where  $\tilde{N}_r(t) \in \mathbb{R}^m$  is defined as

$$\tilde{N}_r \triangleq N_r - \hat{N}_r. \quad (43)$$

By utilizing (35) and (40),  $\tilde{N}_r(t)$  can be rewritten as

$$\tilde{N}_r(t) = \tilde{N}_r(t-T) - k_L r. \quad (44)$$

<sup>8</sup> The expressions in (36), (37) and (41) are based on [20] and [21].

### 3.3 Stability Analysis

**Theorem 3** *The control law (36) and (38) ensures that  $\|e_1(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ , provided that (29) and (30) are satisfied and the elements of  $K$  are selected sufficiently large relative to the system initial conditions (see Appendix C for proof).*

### 3.4 Numerical Simulation Results

A numerical simulation was performed to demonstrate the performance of the learning controller given in (36)-(38). The first order system model in Section 2.4 is utilized with the following reference trajectory

$$x_r = \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} = \begin{bmatrix} \sin(0.2\pi t) \\ \cos(0.2\pi t) \end{bmatrix}. \quad (45)$$

The initial conditions of the system were set to  $x(t_0) = [1 \ -1]^T$ , while the controller parameters were chosen as  $\Lambda = 20I_2$ ,  $K = I_2$ ,  $C_1 = 4I_2$ , and  $k_L = 200$ . The controller parameters above were selected via a trial-error method until a good tracking performance was obtained and then the lower control gain value was preferred. The tracking error  $e_1(t)$  is presented in Figure 4. From Figure 4, it is clear that the tracking objective is satisfied. In Figure 5, the control input  $u(t)$  is presented.

To demonstrate the effect of the learning term  $\hat{W}_r(t)$  in the learning controller in (36), during the simulation run the performance measures in (32) and (33) were computed. For both runs, it was observed that the tracking error converged to zero within 5 seconds. From Table 2, it is clear that after adding the learning term, the controller required less energy while achieving improved tracking performance (*i.e.*, faster convergence).

Table 2  
Comparison of Energy Measures for Learning Control

	$u(t)$ without $\hat{W}_r(t)$	$u(t)$ with $\hat{W}_r(t)$
$M_e$	0.83	0.13
$M_u$	35.14	41.97

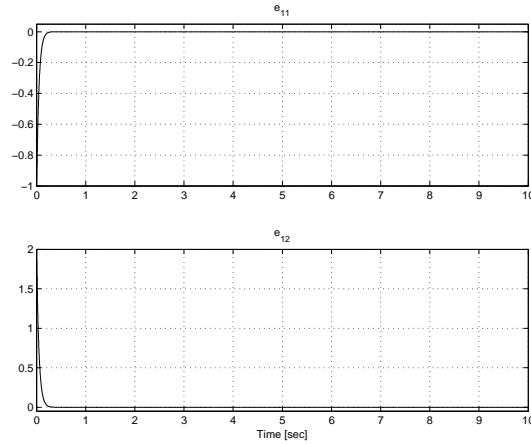


Fig. 4. (Learning Controller) Tracking Error  $e_1(t)$

## 4 Conclusion

Two controllers were developed for a class of MIMO nonlinear systems in the presence of additive disturbances. The robust adaptive controller was proven to yield a semi-global asymptotic tracking result in the presence of parametric

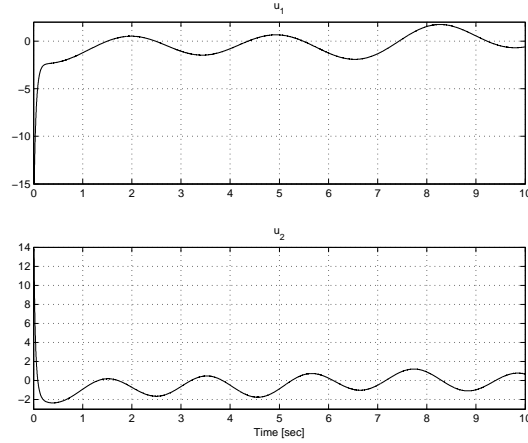


Fig. 5. (Learning Controller) Control Input  $u(t)$

uncertainty along with additive disturbances. The adaptive controller and the adaptation law were designed such that, the parameter estimate vector is proven to go to a constant vector. In the second part of the paper, the learning controller was proven to yield a semi-global asymptotic result in the presence of additive disturbances and when the desired trajectory is periodic. In the development of both controllers, the bounded additive disturbances were assumed to be twice continuously differentiable and have bounded time derivatives up to second order. Since no assumptions were made regarding the periodicity of the disturbances, it is clear that the suggested controllers compensate for both repeating and nonrepeating disturbances. For each controller, Lyapunov-based techniques were used to prove the tracking result. Numerical simulation results were presented for both controllers where nonrepeating disturbances were utilized.

## Acknowledgements

This work is supported in part by a DOC Grant, an ARO Automotive Center Grant, a DOE Contract, a Honda Corporation Grant, and a DARPA Contract.

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## A Proof of Theorem 1

**Lemma 1** *Let the auxiliary functions  $L_1(t)$ ,  $L_2(t) \in \mathbb{R}$  be defined as follows*

$$L_1 \triangleq r^T (\psi - C_1 \text{Sgn}(e_n)) \quad , \quad L_2 \triangleq -\dot{e}_n^T C_2 \text{Sgn}(e_n) . \quad (\text{A.1})$$

*If  $C_1$  is selected to satisfy the sufficient condition (30), then*

$$\int_{t_0}^t L_1(\tau) d\tau \leq \zeta_{b1} \quad , \quad \int_{t_0}^t L_2(\tau) d\tau \leq \zeta_{b2} \quad (\text{A.2})$$

*where  $\zeta_{b1}, \zeta_{b2} \in \mathbb{R}$  are positive constants.*

**PROOF.** After substituting (10) into (A.1) and then integrating  $L_1(t)$  in time, results in the following expression

$$\begin{aligned} \int_{t_0}^t L_1(\tau) d\tau &= \int_{t_0}^t e_n^T(\tau) \Lambda^T [\psi(\tau) \\ &\quad - C_1 \text{Sgn}(e_n(\tau))] d\tau \\ &\quad + \int_{t_0}^t \frac{de_n^T(\tau)}{d\tau} \psi(\tau) d\tau \\ &\quad - \int_{t_0}^t \frac{de_n^T(\tau)}{d\tau} C_1 \text{Sgn}(e_n(\tau)) d\tau . \end{aligned} \quad (\text{A.3})$$

After integrating the second integral on the right-hand side of (A.3) by parts, the following expression is obtained

$$\begin{aligned}
\int_{t_0}^t L_1(\tau) d\tau &= \int_{t_0}^t e_n^T(\tau) \Lambda^T [\psi(\tau) \\
&\quad - C_1 \text{Sgn}(e_n(\tau))] d\tau + e_n^T(\tau) \psi(\tau) \Big|_{t_0}^t \\
&\quad - \int_{t_0}^t e_n^T(\tau) \frac{d\psi(\tau)}{d\tau} d\tau \\
&\quad - \sum_{i=1}^m C_{1i} |e_{ni}(\tau)| \Big|_{t_0}^t \\
&= \int_{t_0}^t e_n^T(\tau) \Lambda^T [\psi(\tau) \\
&\quad - \Lambda^{-1} \frac{d\psi(\tau)}{d\tau} - C_1 \text{Sgn}(e_n(\tau))] d\tau \\
&\quad + e_n^T(t) \psi(t) - e_n^T(t_0) \psi(t_0) \\
&\quad - \sum_{i=1}^m C_{1i} (|e_{ni}(t)| - |e_{ni}(t_0)|). \tag{A.4}
\end{aligned}$$

The right-hand side of (A.4) can be upper-bounded as follows

$$\begin{aligned}
\int_{t_0}^t L_1(\tau) d\tau &\leq \int_{t_0}^t \sum_{i=1}^m |e_{ni}(\tau)| \Lambda_i [|\psi_i(\tau)| \\
&\quad + \frac{1}{\Lambda_i} \left| \frac{d\psi_i(\tau)}{d\tau} \right| - C_{1i}] d\tau \\
&\quad + \sum_{i=1}^m |e_{ni}(t)| (|\psi_i(t)| - C_{1i}) + \zeta_{b1}. \tag{A.5}
\end{aligned}$$

If  $C_1$  is chosen according to satisfy (30), then the first inequality in (A.2) can be proven from (A.5). The second inequality in (A.2) can be obtained by integrating  $L_2(t)$  defined in (A.1) as follows

$$\begin{aligned}
\int_{t_0}^t L_2(\tau) d\tau &= - \int_{t_0}^t e_n^T(\tau) C_2 \text{Sgn}(e_n(\tau)) d\tau \\
&= \zeta_{b2} - \sum_{i=1}^m C_{2i} |e_{ni}(t)| \leq \zeta_{b2}. \tag{A.6}
\end{aligned}$$

■

Let the auxiliary functions  $P_1(t), P_2(t) \in \mathbb{R}$  be defined as follows

$$P_1 \triangleq \zeta_{b1} - \int_{t_0}^t L_1(\tau) d\tau \tag{A.7}$$

$$P_2 \triangleq \zeta_{b2} - \int_{t_0}^t L_2(\tau) d\tau. \tag{A.8}$$

The proof of Lemma 1 ensures that  $P_1(t)$  and  $P_2(t)$  are non-negative. The non-negative function  $V(s(t), t) \in \mathbb{R}$  is defined as follows

$$V \triangleq \frac{1}{2} \sum_{i=1}^n e_i^T e_i + \frac{1}{2} r^T M r + P_1 + P_2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \tag{A.9}$$

where  $s(t) \in \mathbb{R}^{[(n+1)m+2+p] \times 1}$  is defined as

$$s = \begin{bmatrix} z^T & \sqrt{P_1} & \sqrt{P_2} & \tilde{\theta}^T \end{bmatrix}^T. \quad (\text{A.10})$$

By utilizing (5), (A.9) can be bounded<sup>9</sup> as follows

$$W_1(s) \leq V(s, t) \leq W_2(s) \quad (\text{A.11})$$

where  $W_1(s), W_2(s) \in \mathbb{R}$  are defined as

$$W_1(s) \triangleq \lambda_1 \|s\|^2, \quad W_2(s) \triangleq \lambda_2 (\|s\|) \|s\|^2 \quad (\text{A.12})$$

and  $\lambda_1, \lambda_2(\cdot) \in \mathbb{R}$  are defined as

$$\begin{aligned} \lambda_1 &= \frac{1}{2} \min \{1, \underline{m}, \lambda_{\min}(\Gamma^{-1})\}, \\ \lambda_2 &= \max \left\{ 1, \frac{1}{2} \bar{m}(\|s\|), \frac{1}{2} \lambda_{\max}(\Gamma^{-1}) \right\}. \end{aligned} \quad (\text{A.13})$$

The time derivative of (A.9) can be obtained as follows

$$\dot{V} = \sum_{i=1}^n e_i^T \dot{e}_i + r^T \dot{M}r + \frac{1}{2} r^T \dot{M}r + \dot{P}_1 + \dot{P}_2 + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}. \quad (\text{A.14})$$

The first term in the above expression can be written as follows

$$\begin{aligned} \sum_{i=1}^n e_i^T \dot{e}_i &= e_1^T (e_2 - e_1) + e_2^T (e_3 - e_2 - e_1) \\ &\quad + e_3^T (e_4 - e_3 - e_2) + \dots \\ &\quad + e_{n-1}^T (e_n - e_{n-1} - e_{n-2}) + e_n^T (r - \Lambda e_n) \\ &= - \sum_{i=1}^{n-1} e_i^T e_i + e_{n-1}^T e_n + e_n^T r - e_n^T \Lambda e_n \end{aligned} \quad (\text{A.15})$$

where (8a)-(8c), (10) where utilized. Substituting (26), (27), (A.1), and (A.15) into (A.14) results in the following expression

$$\begin{aligned} \dot{V} &= - \sum_{i=1}^{n-1} e_i^T e_i + e_{n-1}^T e_n + e_n^T r - e_n^T \Lambda e_n \\ &\quad + r^T \begin{pmatrix} -\frac{1}{2} \dot{M}r - e_n - (K + I_m) r \\ + W_r \tilde{\theta} - (C_1 + C_2) \text{Sgn}(e_n) + \tilde{N} + \psi \end{pmatrix} \\ &\quad + \frac{1}{2} r^T \dot{M}r - r^T (\psi - C_1 \text{Sgn}(e_n)) \\ &\quad + \dot{e}_n^T C_2 \text{Sgn}(e_n) - \tilde{\theta}^T W_r^T r \end{aligned} \quad (\text{A.16})$$

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<sup>9</sup> Using (6) and (8a)-(8c) it can be shown that  $\|(x, \dot{x}, \dots, x^{(n-1)})\| \leq \vartheta(\|s\|)$  where  $\vartheta(\cdot)$  is some positive function. Thus,  $\bar{m}(x, \dot{x}, \dots, x^{(n-1)}) \leq \bar{m}(\|s\|)$ .



which can be simplified as follows

$$\begin{aligned}\dot{V} = & - \sum_{i=1}^{n-1} e_i^T e_i + e_{n-1}^T e_n - e_n^T \Lambda e_n + \dot{e}_n^T C_2 \text{Sgn}(e_n) \\ & + r^T \left( - (K + I_m) r - C_2 \text{Sgn}(e_n) + \tilde{N} \right)\end{aligned}\quad (\text{A.17})$$

and utilizing (10) results in the following expression

$$\begin{aligned}\dot{V} = & - \sum_{i=1}^{n-1} e_i^T e_i - e_n^T \Lambda e_n + e_{n-1}^T e_n - r^T r \\ & + r^T \tilde{N} - r^T K r - e_n^T \Lambda C_2 \text{Sgn}(e_n).\end{aligned}\quad (\text{A.18})$$

By using (19), (29), and the triangle inequality, an upper-bound on (A.18) can be obtained as follows

$$\begin{aligned}\dot{V} \leq & -\lambda_3 \|z\|^2 + \|r\| \rho(\|z\|) \|z\| \\ & - \underline{K} \|r\|^2 - \sum_{i=1}^m \Lambda_i C_{2i} |e_{ni}(t)| \\ \leq & - \left( \lambda_3 - \frac{\rho^2(\|z\|)}{4\underline{K}} \right) \|z\|^2 - \sum_{i=1}^m \Lambda_i C_{2i} |e_{ni}(t)|\end{aligned}\quad (\text{A.19})$$

where  $\lambda_3 \triangleq \min \left\{ \frac{1}{2}, \lambda_{\min}(\Lambda) - \frac{1}{2} \right\}$  and  $\underline{K} \in \mathbb{R}$  is the minimum eigenvalue of  $K$ . The following inequality can be developed<sup>10</sup>

$$\dot{V} \leq W(s) - \sum_{i=1}^m \Lambda_i C_{2i} |e_{ni}(t)| \quad (\text{A.20})$$

where  $W(s) \in \mathbb{R}$  denotes the non-positive function

$$W(s) \triangleq -\beta_0 \|z\|^2 \quad (\text{A.21})$$

in which  $\beta_0 \in \mathbb{R}$  is a positive constant, and provided that  $\underline{K}$  is selected according to the following sufficient condition

$$\underline{K} \geq \frac{\rho^2(\|z\|)}{4\lambda_3} \text{ or } \|z\| \leq \rho^{-1} \left( 2\sqrt{\lambda_3 \underline{K}} \right). \quad (\text{A.22})$$

Based on (A.9)-(A.13) and (A.19)-(A.21) the regions  $\mathcal{D}$  and  $\mathcal{S}$  can be defined as follows

$$\mathcal{D} = \left\{ s : \|s\| < \rho^{-1} \left( 2\sqrt{\lambda_3 \underline{K}} \right) \right\} \quad (\text{A.23})$$

$$\mathcal{S} = \left\{ s \in \mathcal{D} : W_2(s) < \lambda_1 \left( \rho^{-1} \left( 2\sqrt{\lambda_3 \underline{K}} \right) \right)^2 \right\}. \quad (\text{A.24})$$

Note that the region of attraction in (A.24) can be made arbitrarily large to include any initial conditions by increasing  $\underline{K}$  (*i.e.*, a semi-global stability result). Specifically, (A.12) and (A.24) can be used to calculate the region

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<sup>10</sup> The expression in (A.20) can be rewritten as follows

$$\dot{V} \leq -\underline{\Lambda} \underline{C}_2 \|e_n(t)\|_1$$

where  $\underline{\Lambda}$  and  $\underline{C}_2 \in \mathbb{R}$  are the minimum eigenvalues of  $\Lambda$  and  $C_2$ , respectively. Based on the subsequent analysis, it is clear that  $e_n(t) \in \mathcal{L}_1$ .

of attraction as follows

$$\begin{aligned} W_2(s(t_0)) &< \lambda_1 \left( \rho^{-1} \left( 2\sqrt{\lambda_3 \underline{K}} \right) \right)^2 \\ \implies \|s(t_0)\| &< \sqrt{\frac{\lambda_1}{\lambda_2(\|s(t_0)\|)}} \rho^{-1} \left( 2\sqrt{\lambda_3 \underline{K}} \right) \end{aligned} \quad (\text{A.25})$$

which can be rearranged as

$$\underline{K} \geq \frac{1}{4\lambda_3} \rho^2 \left( \sqrt{\frac{\lambda_2(\|s(t_0)\|)}{\lambda_1}} \|s(t_0)\| \right). \quad (\text{A.26})$$

By utilizing (20) and (A.10) the following explicit expression for  $\|s(t_0)\|$  can be derived as follows

$$\|s(t_0)\|^2 = \sum_{i=1}^n \|e_i(t_0)\|^2 + \|r(t_0)\|^2 + \zeta_{b1} + \zeta_{b2} + \|\theta\|^2.$$

From (A.9), (A.20), (A.24)-(A.26), it is clear that  $V(s, t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}^{11}$ ; hence  $s(t), z(t), \tilde{\theta}(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . From (A.20) it is easy to prove that  $e_n(t) \in \mathcal{L}_1 \forall s(t_0) \in \mathcal{S}$ . From (10), it is clear that  $\dot{e}_n(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . By using (6), (7) and (9), it can be proven that  $x^{(i)}(t) \in \mathcal{L}_\infty, i = 0, 1, \dots, n, \forall s(t_0) \in \mathcal{S}$ . Then, it is clear that  $M(t), \dot{M}(t), f(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . The facts that  $r(t), \tilde{\theta}(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$  can be used along with (28) and (26) to prove that  $\hat{\theta}(t), \dot{\hat{\theta}}(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . After using these boundedness statements along with (2) and (25), it is clear that  $u(t), \dot{u}(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . The previous boundedness statements and Remarks 1, 2, 3 can be used along with (14), to prove that  $\dot{r}(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . These boundedness statements can be used along with the time derivative of (A.21) to prove that  $\dot{W}(s(t)) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ ; hence  $W(s(t))$  is uniformly continuous. Standard signal chasing algorithms can be used to prove that all remaining signals are bounded. A direct application of Theorem 8.4 in [8] can be used to prove that  $\|z(t)\| \rightarrow 0$  as  $t \rightarrow \infty \forall s(t_0) \in \mathcal{S}$ . Based on the definition of  $z(t)$ , it is easy to show that  $\|e_i(t)\|, \|r(t)\| \rightarrow 0$  as  $t \rightarrow \infty \forall s(t_0) \in \mathcal{S}, i = 1, 2, \dots, n$ . From (10), it is clear that  $\|\dot{e}_n(t)\| \rightarrow 0$  as  $t \rightarrow \infty \forall s(t_0) \in \mathcal{S}$ . By utilizing (9) recursively it can be proven that  $\|e_1^{(i)}(t)\| \rightarrow 0$  as  $t \rightarrow \infty, i = 1, 2, \dots, n \forall s(t_0) \in \mathcal{S}$ . ■

## B Proof of Theorem 2

**PROOF.** The fact that  $W_r(t)$  is a function of only  $x_r(t)$  and its time derivatives, can be used along with the boundedness requirement in (7), to show that  $W_r(t), \dot{W}_r(t) \in \mathcal{L}_\infty$ . After considering the fact that  $e_n(t) \in \mathcal{L}_1$  (see the proof of Theorem 1), it is clear that  $W_r^T(t) \Lambda e_n(t), \dot{W}_r^T(t) e_n(t) \in \mathcal{L}_1$ . After utilizing this fact along with the first and second terms in (24), we can conclude that  $\int_{t_0}^t W_r^T(\tau) \Lambda e_n(\tau) d\tau \rightarrow c_1$  and  $\int_{t_0}^t \dot{W}_r^T(\tau) e_n(\tau) d\tau \rightarrow c_2$  as  $t \rightarrow \infty$  where  $c_1, c_2$  are constant vectors (see Theorem 3.1 of [10]). Based on the fact that  $e_n(t) \rightarrow 0_{m \times 1}$  as  $t \rightarrow \infty \forall s(t_0) \in \mathcal{S}$  (see the proof of Theorem 1) then it is clear that  $W_r^T(t) e_n(t) \rightarrow 0_{m \times 1}$  as  $t \rightarrow \infty$ . Utilizing the above facts along with the fact that  $W_r^T(t_0) e_n(t_0)$  is constant, it follows that  $\hat{\theta}(t) \rightarrow \hat{\theta}_\infty$  as  $t \rightarrow \infty$ . ■

## C Proof of Theorem 3

**PROOF.** Let  $V(s, t) \in \mathbb{R}$  denote the following non-negative function

$$V \triangleq \frac{1}{2} \sum_{i=1}^n e_i^T e_i + \frac{1}{2} r^T M r + P_1 + V_g \quad (\text{C.1})$$

<sup>11</sup> Similar steps in [8] and [21] can be utilized to prove that  $s(t) \in \mathcal{S} \forall t$  when  $s(t_0) \in \mathcal{S}$  and (29), (30), and (A.26) are satisfied.

where  $P_1(t)$  was defined in (A.7),  $V_g(t) \in \mathbb{R}$  is a non-negative function defined as

$$V_g \triangleq \frac{1}{2k_L} \int_{t-T}^t \tilde{N}_r^T(\tau) \tilde{N}_r(\tau) d\tau, \quad (\text{C.2})$$

and  $s(t)$  is defined as

$$s \triangleq \begin{bmatrix} z^T & \sqrt{P_1} & \sqrt{V_g} \end{bmatrix}^T. \quad (\text{C.3})$$

After utilizing (5), (C.1) can be bounded as follows

$$W_1(s) \leq V(s, t) \leq W_2(s) \quad (\text{C.4})$$

where  $W_1(s), W_2(s) \in \mathbb{R}$  are defined by

$$W_1(s) \triangleq \lambda_1 \|s\|^2, \quad W_2(s) \triangleq \lambda_2 (\|s\|) \|s\|^2 \quad (\text{C.5})$$

and  $\lambda_1, \lambda_2(\cdot) \in \mathbb{R}$  are defined according to

$$\lambda_1 \triangleq \frac{1}{2} \min \{1, \underline{m}\}, \quad \lambda_2 \triangleq \max \left\{ 1, \frac{1}{2} \bar{m}(\|s\|) \right\}. \quad (\text{C.6})$$

After taking the time derivative of (C.1), the following expression can be obtained

$$\begin{aligned} \dot{V} = & - \sum_{i=1}^{n-1} e_i^T e_i - e_n^T \Lambda e_n + e_{n-1}^T e_n \\ & - r^T r + r^T \tilde{N} - r^T K r - \frac{k_L}{2} r^T r \end{aligned} \quad (\text{C.7})$$

where (8a)-(8c), (10), (42), (44) and (A.1) were utilized. After utilizing (19), (29) and the triangle inequality, an upper-bound on (C.7) can be obtained as

$$\begin{aligned} \dot{V} \leq & -\lambda_3 \|z\|^2 + \|r\| \rho(\|z\|) \|z\| - \left( \underline{K} + \frac{k_L}{2} \right) \|r\|^2 \\ \leq & - \left( \lambda_4 - \frac{\rho^2(\|z\|)}{4\underline{K}} \right) \|z\|^2 \end{aligned} \quad (\text{C.8})$$

where  $\lambda_3 \triangleq \min \left\{ \frac{1}{2}, \lambda_{\min}(\Lambda) - \frac{1}{2} \right\}$  and  $\lambda_4 \triangleq \min \left\{ \lambda_3, \frac{k_L}{2} \right\}$ . The following inequality can be developed

$$\dot{V} \leq W(s) \leq \bar{W}(s) \quad (\text{C.9})$$

where  $W(s), \bar{W}(s) \in \mathbb{R}$  denote the following non-positive functions

$$W(s) \triangleq -\beta_0 \|z\|^2, \quad \bar{W}(s) \triangleq -\beta_0 \|e_1\|^2 \quad (\text{C.10})$$

with  $\beta_0 \in \mathbb{R}$  being a positive constant, and provided that  $\underline{K}$  is selected according to the following sufficient condition

$$\underline{K} \geq \frac{\rho^2(\|z\|)}{4\lambda_4} \text{ or } \|z\| \leq \rho^{-1} \left( 2\sqrt{\lambda_4 \underline{K}} \right). \quad (\text{C.11})$$

Based on (C.1)-(C.6) and (C.8)-(C.10), the regions  $\mathcal{D}$  and  $\mathcal{S}$  can be defined as follows

$$\mathcal{D} = \left\{ s : \|s\| < \rho^{-1} \left( 2\sqrt{\lambda_4 \underline{K}} \right) \right\} \quad (\text{C.12})$$

$$\mathcal{S} = \left\{ s \in \mathcal{D} : W_2(s) < \lambda_1 \left( \rho^{-1} \left( 2\sqrt{\lambda_4 \underline{K}} \right) \right)^2 \right\}. \quad (\text{C.13})$$

Note that the region of attraction in (C.13) can be made arbitrarily large to include any initial conditions by increasing  $\underline{K}$  (i.e., a semi-global stability result). Specifically, (C.5) and (C.13) can be used to calculate the region of attraction as follows

$$\begin{aligned} W_2(s(t_0)) &< \lambda_1 \left( \rho^{-1} \left( 2\sqrt{\lambda_4 \underline{K}} \right) \right)^2 \\ \Rightarrow \|s(t_0)\| &< \sqrt{\frac{\lambda_1}{\lambda_2(\|s(t_0)\|)}} \rho^{-1} \left( 2\sqrt{\lambda_4 \underline{K}} \right) \end{aligned} \quad (\text{C.14})$$

which can be rearranged as

$$\underline{K} \geq \frac{1}{4\lambda_4} \rho^2 \left( \sqrt{\frac{\lambda_2(\|s(t_0)\|)}{\lambda_1}} \|s(t_0)\| \right). \quad (\text{C.15})$$

By utilizing (20) and (C.3) the following explicit expression for  $\|s(t_0)\|$  can be derived as follows

$$\|s(t_0)\|^2 = \sum_{i=1}^n \|e_i(t_0)\|^2 + \|r(t_0)\|^2 + \zeta_{b1}.$$

From (C.1), (C.9), (C.13)-(C.15), it is clear that  $V(s, t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ ; hence  $s(t), z(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . From (10), it is clear that  $\dot{e}_n(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . Using (6) and (34), it can be proved that  $x^{(i)}(t) \in \mathcal{L}_\infty, i = 0, 1, \dots, n, \forall s(t_0) \in \mathcal{S}$ . Then, it is clear that  $M(t), \dot{M}(t), f(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . By using these boundedness statements along with (2) it is clear that  $u(t) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ . These boundedness statements can be used along with the time derivative of (C.10) to prove that  $\dot{W}(s(t)) \in \mathcal{L}_\infty \forall s(t_0) \in \mathcal{S}$ ; hence  $\bar{W}(s(t))$  is uniformly continuous. A direct application of Theorem 8.4 in [8] can be used to prove that  $\|e_1(t)\| \rightarrow 0$  as  $t \rightarrow \infty \forall s(t_0) \in \mathcal{S}$ . It should be noted that for finite time the subsequent analysis can be easily extended to prove that  $\hat{N}_r(t), \dot{u}(t), \dot{r}(t), \tilde{N}_r(t)$  are bounded. ■

## D Development of (11) for a second order system

In this appendix, the open-loop error system in (11) is developed in detail for a second order system. Following system model which is a second order form of the class of nonlinear systems is considered

$$\ddot{x} = f + G(u + d_1) + d_2$$

where  $x(t), \dot{x}(t) \in \mathbb{R}^m$  are the system states,  $f(x, \dot{x}, \theta) \in \mathbb{R}^m$  and  $G(x, \dot{x}, \theta) \in \mathbb{R}^{m \times m}$  are nonlinear functions,  $\theta \in \mathbb{R}^p$  is an unknown constant parameter vector,  $d_1(t), d_2(t) \in \mathbb{R}^m$  are unknown additive nonlinear disturbances, and  $u(t) \in \mathbb{R}^m$  is the control input. The system model is assumed to satisfy the following assumptions.

**Assumption 5** The nonlinear function  $G(\cdot)$  is symmetric, positive definite and satisfies the following inequalities

$$\underline{m} \|\xi\|^2 \leq \xi^T M(\cdot) \xi \leq \bar{m}(\cdot) \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^m$$

where  $M(x, \dot{x}, \theta) \in \mathbb{R}^{m \times m}$  is defined as

$$M \triangleq G^{-1} \quad (\text{D.1})$$

and  $\underline{m} \in \mathbb{R}$  is a positive bounding constant,  $\bar{m}(x, \dot{x}) \in \mathbb{R}$  is a positive, globally invertible, nondecreasing function of each variable, and  $\|\cdot\|$  denotes the Euclidean norm.

**Assumption 6** The nonlinear functions,  $f(\cdot)$  and  $G(\cdot)$ , are continuously differentiable up to their second derivatives (i.e.,  $f(\cdot), G(\cdot) \in \mathcal{C}^2$ ).

**Assumption 7** The nonlinear functions,  $f(\cdot)$  and  $M(\cdot)$ , are affine in  $\theta$ .

**Assumption 8** The additive disturbances,  $d_1(t)$  and  $d_2(t)$ , are assumed to be continuously differentiable and bounded up to their second derivatives (i.e.,  $d_i(t) \in \mathcal{C}^2$  and  $d_i(t), \dot{d}_i(t), \ddot{d}_i(t) \in \mathcal{L}_\infty, i = 1, 2$ ).

The output tracking error  $e_1(t) \in \mathbb{R}^m$  is defined as follows

$$e_1 \triangleq x_r - x \quad (\text{D.2})$$

where  $x_r(t) \in \mathbb{R}^m$  is the reference trajectory satisfying the following property

$$x_r(t) \in \mathcal{C}^2, x_r^{(i)}(t) \in \mathcal{L}_\infty, i = 0, 1, \dots, 4. \quad (\text{D.3})$$

The control design objective is to develop an adaptive control law that ensures  $\|e_1(t)\|, \|\dot{e}_1(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  and that all signals remain bounded within the closed-loop system. To achieve the control objectives, the subsequent development is derived based on the assumption that the system states  $x(t)$  and  $\dot{x}(t)$  are measurable.

The filtered tracking error signal, denoted by  $e_2(t) \in \mathbb{R}^m$ , is defined as follows

$$e_2 \triangleq \dot{e}_1 + e_1. \quad (\text{D.4})$$

After utilizing (D.1), the system model can be rewritten as follows

$$M\ddot{x} = h + u + d_1 + Md_2 \quad (\text{D.5})$$

where  $h(t) \in \mathbb{R}^m$  is defined as follows

$$h \triangleq Mf. \quad (\text{D.6})$$

To facilitate the control development, the filtered tracking error signal, denoted by  $r(t) \in \mathbb{R}^m$ , is defined as follows

$$r \triangleq \dot{e}_2 + \Lambda e_2 \quad (\text{D.7})$$

where  $\Lambda \in \mathbb{R}^{m \times m}$  is a constant, diagonal, positive definite, gain matrix. After differentiating (D.7) following expression is obtained

$$\dot{r} \triangleq \ddot{e}_2 + \Lambda \dot{e}_2 \quad (\text{D.8})$$

$$= \ddot{\ddot{e}}_1 + \ddot{\ddot{e}}_1 + \Lambda \dot{e}_2 \quad (\text{D.9})$$

$$= \ddot{\ddot{x}}_r - \ddot{\ddot{x}} + \ddot{\ddot{e}}_1 + \Lambda \dot{e}_2 \quad (\text{D.10})$$

where second time derivative of (D.4), and third time derivative of (D.2) were utilized to obtain the second and third equations, respectively. The time derivative of the expression in (D.5) is as follows

$$M\ddot{\ddot{x}} + \dot{M}\ddot{x} = \dot{h} + \dot{u} + \dot{d}_1 + \dot{M}d_2 + M\dot{d}_2 \quad (\text{D.11})$$

and the following expression is obtained

$$M\ddot{\ddot{x}} = -\dot{M}\ddot{x} + \dot{h} + \dot{u} + \dot{d}_1 + \dot{M}d_2 + M\dot{d}_2. \quad (\text{D.12})$$

After premultiplying (D.10) by  $M(\cdot)$  following expression is obtained

$$M\dot{r} = M\ddot{\ddot{x}}_r - M\ddot{\ddot{x}} + M\ddot{\ddot{e}}_1 + M\Lambda\dot{e}_2 \quad (\text{D.13})$$

and then substituting (D.12) results in the following expression

$$M\dot{r} = M\ddot{\ddot{x}}_r + M\ddot{\ddot{e}}_1 + M\Lambda\dot{e}_2 + \dot{M}\ddot{x} - \dot{h} - \dot{u} - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 \quad (\text{D.14})$$

$$= M(\ddot{\ddot{x}}_r + \ddot{\ddot{e}}_1 + \Lambda\dot{e}_2) + \dot{M}\ddot{x} - \dot{h} - \dot{u} - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2. \quad (\text{D.15})$$

It is noted that (D.15) is the second order form of the general expression in (11).

## E Detailed explanation of the system model in the simulation study

In this appendix, the system model utilized in the simulation study is presented in detail. The system model that was considered in the simulation study is a first order form of the class of nonlinear systems considered in this work and has the following form

$$\dot{x} = f + G(u + d_1) + d_2 \quad (\text{E.1})$$

where  $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T \in \mathbb{R}^2$  is the system state,  $u(t) \in \mathbb{R}^2$  is the control input,  $f(x, \theta) \in \mathbb{R}^2$  and  $G(x, \theta) \in \mathbb{R}^{2 \times 2}$  are nonlinear functions,  $\theta = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^T \in \mathbb{R}^2$  is a constant parameter vector,  $d_1(t), d_2(t) \in \mathbb{R}^2$  are additive nonlinear disturbances, and  $f(x, \theta)$  and  $G(x, \theta)$  are defined as follows

$$f = \begin{bmatrix} x_1 x_2 \\ x_2^2 \end{bmatrix} \quad (\text{E.2})$$

$$G = \begin{bmatrix} \frac{2 + \cos x_1}{\theta_1} & 0 \\ 0 & \frac{3 + \sin x_2}{\theta_2} \end{bmatrix}. \quad (\text{E.3})$$

The matrix inverse of  $G(x, \theta)$ , denoted by  $M(x, \theta) \in \mathbb{R}^{2 \times 2}$ , is defined as

$$M = \begin{bmatrix} \frac{\theta_1}{2 + \cos x_1} & 0 \\ 0 & \frac{\theta_2}{3 + \sin x_2} \end{bmatrix}. \quad (\text{E.4})$$

The output tracking error, denoted by  $e_1(t) \in \mathbb{R}^2$ , is defined as follows

$$e_1 \triangleq x_r - x \quad (\text{E.5})$$

where  $x_r(t) \in \mathbb{R}^2$  is the reference trajectory selected as

$$x_r = \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} = \begin{bmatrix} \sin t \left( 1 - \exp\left(-\frac{t^3}{5}\right) \right) \\ 2 \sin t \left( 1 - \exp\left(-\frac{t^3}{2}\right) \right) \end{bmatrix}. \quad (\text{E.6})$$

After utilizing  $M(\cdot) = G^{-1}(\cdot)$  the system model can be rewritten as follows

$$M\dot{x} = h + u + d_1 + Md_2 \quad (\text{E.7})$$

where  $h(t) \in \mathbb{R}^2$  is defined as follows

$$h \triangleq Mf. \quad (\text{E.8})$$

The time derivative of the expression in (E.7) is given as follows

$$M\ddot{x} + \dot{M}\dot{x} = \dot{h} + \dot{u} + \dot{d}_1 + \dot{M}d_2 + M\dot{d}_2. \quad (\text{E.9})$$

Since a first order system model was preferred for the numerical simulation study, the only filtered tracking error signal, denoted by  $r(t) \in \mathbb{R}^2$ , is defined as follows

$$r \triangleq \dot{e}_1 + \Lambda e_1 \quad (\text{E.10})$$

where  $\Lambda \in \mathbb{R}^{2 \times 2}$  is a constant, diagonal, positive definite, gain matrix. After differentiating (E.10) following expression is obtained

$$\dot{r} \triangleq \ddot{e}_1 + \Lambda \dot{e}_1 \quad (\text{E.11})$$

$$= \ddot{x}_r - \ddot{x} + \Lambda \dot{e}_1 \quad (\text{E.12})$$

where the second time derivative of (E.5) was utilized. After premultiplying (E.12) with  $M(\cdot)$ , following expression is obtained

$$M\dot{r} = M\ddot{x}_r - M\ddot{x} + M\Lambda\dot{e}_1 \quad (\text{E.13})$$

and then substituting (E.9) results in the following expression

$$M\dot{r} = M\ddot{x}_r + \dot{M}\dot{x} - \dot{h} - \dot{u} - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 + M\Lambda\dot{e}_1. \quad (\text{E.14})$$

After adding and subtracting  $0.5\dot{M}(\cdot)r(t) + e_1(t)$  to the right-hand-side of (E.14) following expression is obtained

$$M\dot{r} = M\ddot{x}_r + \dot{M}\dot{x} - \dot{h} - \dot{u} - \dot{d}_1 - M\dot{d}_2 - \dot{M}d_2 + M\Lambda\dot{e}_1 + 0.5\dot{M}r - 0.5\dot{M}r + e_1 - e_1. \quad (\text{E.15})$$

The auxiliary signal, denoted by  $N(\cdot) \in \mathbb{R}^2$ , is defined as follows

$$N = M(\ddot{x}_r + \Lambda\dot{e}_1) + \dot{M}(\dot{x} + 0.5r) + e_1 - \dot{h} \quad (\text{E.16})$$

and the auxiliary signal, denoted by  $N_r(t) \in \mathbb{R}^2$ , is defined as follows

$$N_r = M_r\ddot{x}_r + \dot{M}_r\dot{x}_r - \dot{h}_r \quad (\text{E.17})$$

where  $h_r(t) \in \mathbb{R}^2$  is defined as follows

$$h_r \triangleq M_r f_r. \quad (\text{E.18})$$

The nonlinear functions  $M_r(t)$  and  $f_r(t)$  are functions of reference trajectory and are defined as follows

$$M_r = \begin{bmatrix} \frac{\theta_1}{2 + \cos x_{r1}} & 0 \\ 0 & \frac{\theta_2}{3 + \sin x_{r2}} \end{bmatrix} \quad (\text{E.19})$$

$$f_r = \begin{bmatrix} x_{r1}x_{r2} \\ x_{r2}^2 \end{bmatrix}. \quad (\text{E.20})$$

The auxiliary signal  $N_r(t)$  can be found as follows

$$\begin{aligned} N_r &= \begin{bmatrix} \theta_1 \frac{1}{2 + \cos x_{r1}} & 0 \\ 0 & \theta_2 \frac{1}{3 + \sin x_{r2}} \end{bmatrix} \begin{bmatrix} \ddot{x}_{r1} \\ \ddot{x}_{r2} \end{bmatrix} \\ &+ \begin{bmatrix} -\theta_1 \frac{\dot{x}_{r1} \sin x_{r1}}{(2 + \cos x_{r1})^2} & 0 \\ 0 & \theta_2 \frac{\dot{x}_{r2} \cos x_{r2}}{(3 + \sin x_{r2})^2} \end{bmatrix} \begin{bmatrix} \dot{x}_{r1} \\ \dot{x}_{r2} \end{bmatrix} \\ &- \begin{bmatrix} -\theta_1 \frac{\dot{x}_{r1} \sin x_{r1}}{(2 + \cos x_{r1})^2} & 0 \\ 0 & \theta_2 \frac{\dot{x}_{r2} \cos x_{r2}}{(3 + \sin x_{r2})^2} \end{bmatrix} \begin{bmatrix} x_{r1}x_{r2} \\ x_{r2}^2 \end{bmatrix} \\ &- \begin{bmatrix} \theta_1 \frac{1}{2 + \cos x_{r1}} & 0 \\ 0 & \theta_2 \frac{1}{3 + \sin x_{r2}} \end{bmatrix} \begin{bmatrix} \dot{x}_{r1}x_{r2} + x_{r1}\dot{x}_{r2} \\ 2x_{r2}\dot{x}_{r2} \end{bmatrix} \\ &= \begin{bmatrix} \theta_1 \left( \frac{\ddot{x}_{r1} - \dot{x}_{r1}x_{r2} - x_{r1}\dot{x}_{r2}}{2 + \cos x_{r1}} - \frac{\dot{x}_{r1}(\dot{x}_{r1} - x_{r1}x_{r2}) \sin x_{r1}}{(2 + \cos x_{r1})^2} \right) \\ \theta_2 \left( \frac{\ddot{x}_{r2} - 2x_{r2}\dot{x}_{r2}}{3 + \sin x_{r2}} + \frac{\dot{x}_{r2}(\dot{x}_{r2} - x_{r2}^2) \cos x_{r2}}{(3 + \sin x_{r2})^2} \right) \end{bmatrix} \end{aligned} \quad (\text{E.21})$$

From which it can be concluded that

$$N_r = W_r \theta$$

where  $\theta = [\theta_1 \ \theta_2]^T \in \mathbb{R}^2$  is the unknown constant parameter vector and  $W_r(x_r, \dot{x}_r, \ddot{x}_r) \in \mathbb{R}^{2 \times 2}$  is the nonlinear regressor matrix defined as follows

$$W_r = \begin{bmatrix} \frac{\ddot{x}_{r1} - \dot{x}_{r1}x_{r2} - x_{r1}\dot{x}_{r2}}{2 + \cos x_{r1}} - \frac{\dot{x}_{r1}(\dot{x}_{r1} - x_{r1}x_{r2})\sin x_{r1}}{(2 + \cos x_{r1})^2} & 0 \\ 0 & \frac{\ddot{x}_{r2} - 2x_{r2}\dot{x}_{r2}}{3 + \sin x_{r2}} + \frac{\dot{x}_{r2}(\dot{x}_{r2} - x_{r2}^2)\cos x_{r2}}{(3 + \sin x_{r2})^2} \end{bmatrix}.$$

## F Development of (19)

In this appendix, the upper bound for the auxiliary term  $\tilde{N}(\cdot)$  in ((19)) will be derived. To facilitate the upper bound development, first, the second and the third terms in the first line of (13) will be written in terms of  $e_i(t)$ ,  $i = 1, \dots, n$ . After utilizing (8a) and (8b), following expressions can be obtained for the first and the second time derivatives of  $e_1(t)$

$$\dot{e}_1 = e_2 - e_1 \tag{F.1}$$

$$\ddot{e}_1 = e_3 - 2e_2 \tag{F.2}$$

and the following general formula can be utilized to calculate the higher order derivatives of  $e_1(t)$  in terms of  $e_i(t)$ ,  $i = 1, \dots, n$

$$e_1^{(k)} = \bar{a}_{k,0}e_{k+1} + \bar{a}_{k,1}e_k + \bar{a}_{k,2}e_{k-1} + \sum_{i=1}^{k-2} (-\bar{a}_{k-1,i} - \bar{a}_{k-1,i+1} + \bar{a}_{k-1,i+2})e_{k-1-i} \tag{F.3}$$

for  $k = 3, \dots, (n-1)$ , and the  $n^{\text{th}}$  order derivative of  $e_1(t)$  can be obtained as follows in terms of  $e_i(t)$ ,  $i = 1, \dots, n$  and  $r(t)$

$$e_1^{(n)} = r - [(\bar{a}_{n,0} + \bar{a}_{n,1})I - \Lambda]e_n + (\bar{a}_{n,0} + \bar{a}_{n,2})e_{n-1} + \sum_{i=1}^{n-2} (-\bar{a}_{n-1,i} - \bar{a}_{n-1,i+1} + \bar{a}_{n-1,i+2})e_{n-1-i} \tag{F.4}$$

where  $\bar{a}_{k,0} = 1$ ,  $\bar{a}_{k,1} = -k$ ,  $\bar{a}_{k,2} = (k-1)(k-2)/2$  for  $k = 2, \dots, n$  and  $\bar{a}_{k,j} = 0$  when  $j > k$ . After utilizing (F.3) and (F.4) along with (13), following expression can be obtained

$$\sum_{j=0}^{n-2} a_{nj}e_1^{(j+2)} + \Lambda \dot{e}_n = \sum_{j=1}^{n-1} b_j e_j + \Lambda_1 e_n + \Lambda_2 r \tag{F.5}$$

where  $b_j$ ,  $j = 1, \dots, (n-1)$  are constants, and  $\Lambda_1, \Lambda_2 \in \mathbb{R}^{m \times m}$  are constant, diagonal matrices that can be obtained by substituting (F.1)-(F.4) into the left-hand-side of (F.5). Thus, the auxiliary function  $N(x, \dot{x}, \dots, x^{(n)}, t)$ , which was defined in (13), can be rewritten as follows

$$N = M \left( x_r^{(n+1)} + \sum_{j=1}^{n-1} b_j e_j + \Lambda_1 e_n + \Lambda_2 r \right) + \dot{M} \left( x^{(n)} + \frac{1}{2}r \right) + e_n - \dot{h}. \tag{F.6}$$

The auxiliary function  $N_r(t)$  can be written as follows

$$N_r = M_r x_r^{(n+1)} + \dot{M}_r x_r^{(n)} - \dot{h}_r \tag{F.7}$$



where  $M_r(t)$  was defined in (18) and  $\dot{h}_r(t) \in \mathbb{R}^m$  is defined as follows

$$\dot{h}_r \triangleq \dot{h} \Big|_{x=x_r, \dot{x}=\dot{x}_r, \dots, x^{(n)}=x_r^{(n)}}.$$

To simplify the subsequent derivations, following definitions are made

$$\begin{aligned} F &\triangleq N - M\dot{d}_2 - \dot{M}d_2 \\ F_r &\triangleq N_r - M_r\dot{d}_2 - \dot{M}_rd_2 \end{aligned}$$

where  $F \triangleq F(x, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) \in \mathbb{R}^m$  and  $F_r \triangleq F|_{x=x_r, \dot{x}=\dot{x}_r, \dots, x^{(n)}=x_r^{(n)}} = F(x_r, \dots, x_r^{(n)}, 0, \dots, 0, 0, x_r^{(n+1)}) \in \mathbb{R}^m$ . Thus, from (15) following expression is obtained

$$\tilde{N} \triangleq F - F_r. \tag{F.8}$$

To further facilitate the subsequent upper bound development,  $F(x_r, \dot{x}, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)})$ , ...,  $F(x_r, \dot{x}_r, \dots, x_r^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)})$ ,  $F(x_r, \dots, x_r^{(n)}, 0, \dots, e_n, r, x_r^{(n+1)})$ , ...,  $F(x_r, \dots, x_r^{(n)}, 0, \dots, 0, r, x_r^{(n+1)})$  are added and subtracted to the right-hand side of (F.8) to obtain the following expression

$$\begin{aligned} \tilde{N} &= \left[ F(x, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) - F(x_r, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) \right] \\ &\quad + \left[ F(x_r, \dot{x}, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) - F(x_r, \dot{x}_r, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) \right] \\ &\quad \vdots \\ &\quad + \left[ F(x_r, \dots, x_r^{(n-1)}, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) - F(x_r, \dots, x_r^{(n-1)}, x_r^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) \right] \\ &\quad + \left[ F(x_r, \dots, x_r^{(n-1)}, x_r^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)}) - F(x_r, \dots, x_r^{(n)}, 0, \dots, e_n, r, x_r^{(n+1)}) \right] \\ &\quad + \left[ F(x_r, \dots, x_r^{(n)}, 0, e_2, \dots, e_n, r, x_r^{(n+1)}) - F(x_r, \dots, x_r^{(n)}, 0, 0, \dots, e_n, r, x_r^{(n+1)}) \right] \\ &\quad \vdots \\ &\quad + \left[ F(x_r, \dots, x_r^{(n)}, 0, 0, \dots, 0, r, x_r^{(n+1)}) - F(x_r, \dots, x_r^{(n)}, 0, 0, \dots, 0, 0, x_r^{(n+1)}) \right]. \end{aligned} \tag{F.9}$$

It is to be noted that the first term on the first line of (F.9) and the second term on the last line of (F.9) are equal to  $F(\cdot)$  and  $F_r(\cdot)$ , respectively, and the other terms are subtracted in the preceding line and added in the following line. After applying the Mean Value Theorem [8] to each bracketed term of (F.9), the following expression can be

obtained

$$\begin{aligned}
\tilde{N} = & \frac{\partial F \left( \sigma_0, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)} \right)}{\partial \sigma_0} \Bigg|_{\sigma_0=v_0} (x - x_r) \\
& + \frac{\partial F \left( x_r, \sigma_1, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)} \right)}{\partial \sigma_1} \Bigg|_{\sigma_1=v_1} (\dot{x} - \dot{x}_r) \\
& \vdots \\
& + \frac{\partial F \left( x_r, \dots, \sigma_n, e_1, e_2, \dots, e_n, r, x_r^{(n+1)} \right)}{\partial \sigma_n} \Bigg|_{\sigma_n=v_n} \left( x^{(n)} - x_r^{(n)} \right) \\
& + \frac{\partial F \left( x_r, \dots, x_r^{(n)}, \sigma_{n+1}, e_2, \dots, e_n, r, x_r^{(n+1)} \right)}{\partial \sigma_{n+1}} \Bigg|_{\sigma_{n+1}=v_{n+1}} (e_1 - 0) \\
& + \frac{\partial F \left( x_r, \dots, x_r^{(n)}, e_1, \sigma_{n+2}, \dots, e_n, r, x_r^{(n+1)} \right)}{\partial \sigma_{n+2}} \Bigg|_{\sigma_{n+2}=v_{n+2}} (e_2 - 0) \\
& \vdots \\
& + \frac{\partial F \left( x_r, \dots, x_r^{(n)}, e_1, e_2, \dots, \sigma_{2n}, r, x_r^{(n+1)} \right)}{\partial \sigma_{2n}} \Bigg|_{\sigma_{2n}=v_{2n}} (e_n - 0) \\
& + \frac{\partial F \left( x_r, \dots, x_r^{(n)}, e_1, e_2, \dots, e_n, \sigma_{2n+1}, x_r^{(n+1)} \right)}{\partial \sigma_{2n+1}} \Bigg|_{\sigma_{2n+1}=v_{2n+1}} (r - 0)
\end{aligned} \tag{F.10}$$

where  $v_i \in (x_r, x)$  for  $i = 0, \dots, n$ ,  $v_i \in (0, e_i)$  for  $i = (n+1), \dots, 2n$ , and  $v_{2n+1} \in (0, r)$ . The right-hand side of (F.10) can be upper bounded as follows

$$\begin{aligned}
\|\tilde{N}\| \leq & \left\| \frac{\partial F(\sigma_0, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)})}{\partial \sigma_0} \right\|_{\sigma_0=v_0} \|e_1\| \\
& + \left\| \frac{\partial F(x_r, \sigma_1, \dots, x^{(n)}, e_1, \dots, e_n, r, x_r^{(n+1)})}{\partial \sigma_1} \right\|_{\sigma_1=v_1} \|\dot{e}_1\| \\
& \vdots \\
& + \left\| \frac{\partial F(x_r, \dots, \sigma_n, e_1, e_2, \dots, e_n, r, x_r^{(n+1)})}{\partial \sigma_n} \right\|_{\sigma_n=v_n} \|e_1^{(n)}\| \\
& + \left\| \frac{\partial F(x_r, \dots, x_r^{(n)}, \sigma_{n+1}, e_2, \dots, e_n, r, x_r^{(n+1)})}{\partial \sigma_{n+1}} \right\|_{\sigma_{n+1}=v_{n+1}} \|e_1\| \\
& + \left\| \frac{\partial F(x_r, \dots, x_r^{(n)}, e_1, \sigma_{n+2}, \dots, e_n, r, x_r^{(n+1)})}{\partial \sigma_{n+2}} \right\|_{\sigma_{n+2}=v_{n+2}} \|e_2\| \\
& \vdots \\
& + \left\| \frac{\partial F(x_r, \dots, x_r^{(n)}, e_1, e_2, \dots, \sigma_{2n}, r, x_r^{(n+1)})}{\partial \sigma_{2n}} \right\|_{\sigma_{2n}=v_{2n}} \|e_n\| \\
& + \left\| \frac{\partial F(x_r, \dots, x_r^{(n)}, e_1, e_2, \dots, e_n, \sigma_{2n+1}, x_r^{(n+1)})}{\partial \sigma_{2n+1}} \right\|_{\sigma_{2n+1}=v_{2n+1}} \|r\|. \tag{F.11}
\end{aligned}$$

The partial derivatives in (F.11) can be calculated by using (F.6) as follows

$$\frac{\partial F(\sigma_i)}{\partial \sigma_i} = \frac{\partial M}{\partial \sigma_i} \left( x_r^{(n+1)} + \sum_{j=1}^{n-1} b_j e_j + \Lambda_1 e_n + \Lambda_2 r \right) + \frac{\partial \dot{M}}{\partial \sigma_i} \left( x^{(n)} + \frac{1}{2} r \right) - \frac{\partial \dot{h}}{\partial \sigma_i} \text{ for } i = 0, \dots, n \tag{F.12}$$

$$\frac{\partial F(\sigma_i)}{\partial \sigma_i} = b_i M \text{ for } i = (n+1), \dots, (2n-1) \tag{F.13}$$

$$\frac{\partial F(\sigma_{2n})}{\partial \sigma_{2n}} = M \Lambda_1 + I \tag{F.14}$$

$$\frac{\partial F(\sigma_{2n+1})}{\partial \sigma_{2n+1}} = M \Lambda_2 + 0.5 \dot{M}. \tag{F.15}$$

By defining  $v_i \triangleq x^{(i)} - \tau_i (x^{(i)} - x_r^{(i)})$  for  $i = 0, \dots, n$ ,  $v_i \triangleq e_i - \tau_i (e_i - 0)$  for  $i = (n+1), \dots, 2n$ , and  $v_{2n+1} \triangleq r - \tau_i (r - 0)$ , where  $\tau_i \in (0, 1) \forall i = 0, \dots, (2n+1)$ , and if Assumptions 2 and 4, and (7) are met, then upper bounds for the right-hand sides of the expressions in (F.12)-(F.15) can be rewritten as follows

$$\left\| \frac{\partial F(\sigma_i)}{\partial \sigma_i} \right\|_{\sigma_i=v_i} \leq \rho_i(x, \dots, x^{(n)}) \tag{F.16}$$

where  $\rho_i(\cdot) \forall i = 0, \dots, (2n+1)$ , are positive nondecreasing functions of  $x(t), \dots, x^{(n)}(t)$ . After substituting (F.16) into (F.11), the upper bound for the auxiliary signal  $\tilde{N}(\cdot)$  can be rewritten as

$$\begin{aligned} \tilde{N} \leq & \sum_{j=0}^n \rho_j(\|e_1\|, \dots, \|e_n\|, \|r\|) \left\| e_1^{(j)} \right\| \\ & + \sum_{j=n+1}^{2n} \rho_j(\|e_1\|, \dots, \|e_n\|, \|r\|) \|e_{j-n}\| \\ & + \rho_{2n+1}(\|e_1\|, \dots, \|e_n\|, \|r\|) \|r\| \end{aligned} \quad (\text{F.17})$$

where (F.1)-(F.4) and derivatives of (6) were utilized. The expressions in (20), (F.1)-(F.4) can be used to rewrite the upper bound for the right-hand side of (F.17) as in (19).